

Stably Stratified Shear Turbulence: A New Model for the Energy Dissipation Length Scale

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ABSTRACT

A model is presented to compute the turbulent kinetic energy dissipation length scale l_ϵ in a stably stratified shear flow. The expression for l_ϵ is derived from solving the spectral balance equation for the turbulent kinetic energy. The buoyancy spectrum entering such equation is constructed using a Lagrangian timescale with modifications due to stratification. The final result for l_ϵ is given in algebraic form as a function of the Froude number Fr and the flux Richardson number R_f , $l_\epsilon = l_\epsilon(Fr, R_f)$. The model predicts that for $R_f < R_{fc}$, l_ϵ decreases with stratification or shear; for $R_f > R_{fc}$, which may occur in subgrid-scale models, l_ϵ increases with stratification. An attractive feature of the present model is that it encompasses, as special cases, some seemingly different models for l_ϵ that have been proposed in the past by Deardorff, Hunt et al., Weinstock, and Canuto and Minotti. An alternative form for the dissipation rate ϵ is also discussed that may be useful when one uses a prognostic equation for the heat flux. The present model is applicable to subgrid-scale models, which are needed in large eddy simulations (LES), as well as to ensemble average models.

The model is applied to predict the variation of l_ϵ with height z in the planetary boundary layer. The resulting l_ϵ versus z profile reproduces very closely the nonmonotonic profile of l_ϵ exhibited by many LES calculations, beginning with the one by Deardorff in 1974.

1. Introduction

The availability of large computational facilities has provided a new tool to study turbulent flow, the LES (large eddy simulation), whose conceptual basis rests on the fact that large eddies contain most of the energy, do most of the transporting, are diffusive, anisotropic, long-lived, inhomogeneous, ordered, and dependent on the boundary conditions (Schumann 1991)—qualifications that make it difficult to model them analytically, thus the suggestion to treat them numerically. However, since the number of grid points (or degrees of freedom) N of a turbulent flow, characterized by a Reynolds number Re , grows as $N \sim Re^{9/4}$, it is not possible to resolve all the scales. Unresolved scales smaller than the smallest resolved scale Δ must be modeled. Since small scales are viewed as dissipative, isotropic, short-lived, homogeneous, random, and universal, they are thought to be more amenable to theoretical modeling. Thus, the fusion of an LES with an SGS (subgrid scale) model is considered to be a powerful new tool. This expectation has been largely fulfilled (e.g., Moeng and Wyngaard 1989; Schmidt and

Schumann 1989; Mason and Derbyshire 1990; Nieuwstadt 1990). In the case of stable stratification, however, where the physics differs considerably from the case of unstable stratification, there are still severe doubts as to whether an SGS model is actually available, or, to be more precise, as to whether the physical features of the unresolved scales as envisaged above are correct. For example, if one considers that negative buoyancy strongly hinders vertical motion, one must question the assumption of isotropy. Similar doubts apply to other postulated properties of the unresolved scales.

What is required in an LES are the SGS Reynolds stresses and the SGS convective fluxes

$$\overline{u_i u_j} \quad \text{and} \quad \overline{u_i \theta}, \quad (1)$$

where u_i and θ represent the fluctuating parts of the subgrid-scale velocity and potential temperature fields, respectively. Differential equations for the variables (1) can be constructed using the Reynolds stress models, and a form of these equations is presented in appendix A of Canuto and Minotti (1993, cited as CM). The dynamic equations for the second-order moments (1) imply third-order moments for which one can derive and solve the corresponding dynamic equations, as indeed has been done (e.g., Andre et al. 1982; Canuto et al. 1994). To fix the ideas, let us write the functional dependence of the SGS functions (1) as

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$$\begin{aligned}\overline{u_i u_j} &= f(S_{ij}, R_{ij}, \beta_i | e, \epsilon) \\ \overline{u_i \theta} &= g(S_{ij}, R_{ij}, \beta_i | e, \epsilon),\end{aligned}\quad (2)$$

where we have defined

$$2S_{ij} = \frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}, \quad 2R_{ij} = \frac{\partial U_i}{\partial x_j} - \frac{\partial U_j}{\partial x_i}, \quad \beta_i = \frac{\partial T}{\partial x_i}. \quad (3)$$

Here S_{ij} , R_{ij} , and β_i are the strain rate, the vorticity, and the temperature gradient of the large, resolved scales and thus are known quantities, while e (turbulent kinetic energy) and ϵ (the rate of dissipation of kinetic energy) are turbulence variables that characterize the unresolved scales. Only when the latter are expressed in terms of the large scales will the SGS model be complete. In engineering studies, it has been common practice for many years to employ two differential equations for e and ϵ , the so-called $K-\epsilon$ model (K rather than e is used as a symbol for the turbulent kinetic energy). Could the same philosophy be applied here? The differential equation for e [see CM, Eq. (6a)] presents no conceptual problems, since it can be derived from the basic Navier–Stokes equations (together with a reliable formulation for the third-order moments). The real difficulty is the description of ϵ , the rate of dissipation of turbulent kinetic energy. While there is an exact dynamic equation for ϵ (Davidov 1961; Speziale 1991), it has been of little practical use since most of the terms entering the equation are difficult even to interpret physically, let alone to evaluate or model. Such an equation has therefore never been solved as such; rather, it has been used as a guide to construct a differential equation for ϵ with adjustable parameters, which is part of the aforementioned $K-\epsilon$ model [Speziale 1991, Eqs. (62c,d)]. In addition to its phenomenological nature, this equation has thus far been tested only for unstable or, more often, neutral stratification. This may have discouraged many people who have thus preferred to suggest a variety of phenomenological expressions for ϵ (Deardorff 1980; Weinstock 1981; Hunt et al. 1988, 1989). The way these suggestions are usually made is not directly via a formula for ϵ but equivalently, through a length scale that is introduced in the following way. The simplest model for (1) is the Kolmogorov–Prandtl–Boussinesq model, whereby one simplifies (1) to the form

$$\begin{aligned}\overline{u_i u_j} &= f(S_{ij}, 0, 0 | e, \epsilon) \\ \overline{u_i \theta} &= g(0, 0, \beta_i | e, \epsilon).\end{aligned}\quad (4)$$

On the basis of dimensional arguments (see, however, CM, appendix C), Eqs. (4) yield

$$\overline{u_i u_j} = -2\nu_t S_{ij}, \quad \overline{u_i \theta} = -\chi_t \beta_i, \quad (5)$$

where the turbulent viscosity ν_t is a combination of the variables e and ϵ :

$$\nu_t = C_\mu \frac{e^2}{\epsilon}, \quad (6a)$$

which is the formula widely employed in the $K-\epsilon$ model. It is important to notice that second-order closure models predict for the coefficient C_μ the value

$$C_\mu = 0.096-0.112, \quad (6b)$$

close to the values suggested empirically by Rodi (1984) and Schumann (1991). The same models also predict an expression for the turbulent conductivity χ_t of a form similar to (6a) with a value of $C_\chi = 0.178$, which is close to the empirical value $C_\chi = 0.172$ (Schumann 1991).

To the extent that one accepts the simplified forms (4) and (5), Eq. (6a) is exact. The important point to notice is that one has to deal with two independent variables, be they

$$(e, \epsilon), (e, \tau), (e, \tau^{-1}), \quad \text{or} \quad (e^{1/2}, \tau^{-1}), \quad (7)$$

where τ is a turbulence timescale $\sim e/\epsilon$. Recently, Lang and Shih (1991) have discussed which pair of variables in (7) is more suitable to describe an unstratified turbulent flow. Their conclusion is that the variables (e, ϵ) are the more robust variables in the sense that the differential equations describing them have a wider degree of applicability requiring the least number of changes of the adjustable parameters when considering different types of flows.

In the geophysical literature, the form (6a) is not usually adopted. Rather, it has been customary to employ the variables (e, l_ϵ) , where the turbulence length scale l_ϵ is defined as

$$l_\epsilon = \frac{e^{3/2}}{\epsilon}. \quad (8a)$$

Since the differential equation for e presents no conceptual problems, the discussion centers on the variable l_ϵ . We may notice that in terms of (e, l_ϵ) ,

$$\nu_t = C_\mu e^{1/2} l_\epsilon. \quad (8b)$$

From time to time, there have been suggestions to translate the differential equation for ϵ into a differential equation for the length scale l_ϵ , the work of Rotta (1951) being the first on the subject. However, the lack of guidance as to how to include stable stratification into these equations affects both ϵ and l_ϵ equally.

How can one derive a model for l_ϵ ? Let $E(k)$ be the turbulent kinetic energy spectral function, defined so that the kinetic energy e of the subgrid scales (with wavenumbers larger than the maximum resolved wavenumber $k_m = \pi/\Delta$) is given by

$$e = \int_{\pi/\Delta}^{\infty} E(k) dk, \quad (9)$$

and let us assume that $E(k)$ has a quasi-inertial form (Lumley 1964)

$$E(k) = \text{Ko}\epsilon(k)^{2/3}k^{-5/3}. \quad (10)$$

In a purely inertial region, $\epsilon(k)$ is, by Kolmogorov assumption, independent of k . In stably stratified situations, this is no longer true, and thus we write, still formally,

$$\epsilon(k) \equiv f(k, S, B|\epsilon), \quad (11a)$$

where S and B represent the dependence on shear and buoyancy, while

$$\epsilon \equiv \epsilon(k \rightarrow \infty) \quad (11b)$$

is the true dissipation rate of turbulent kinetic energy. If one could construct the function (11a), use of it in (10) and then in (9) would lead to a relation of the form

$$e = e(S, B, \Delta|\epsilon), \quad (12a)$$

which could be cast in the form of Eq. (8a) for ϵ so as to finally obtain

$$l_\epsilon = l_\epsilon(S, B, \Delta|e), \quad (12b)$$

or, more compactly,

$$l_\epsilon = l_\epsilon(\text{Fr}, \text{Sh}), \quad (13a)$$

where Sh and Fr are the shear and Froude numbers, respectively,

$$\text{Sh} = \frac{\Delta S}{e^{1/2}}, \quad \text{Fr} = \frac{e^{1/2}}{\Delta N}, \quad (13b)$$

where

$$S^2 = 2S_{ij}S_{ij}, \quad N = \left(g_i \alpha \frac{\partial T}{\partial x_i} \right)^{1/2}. \quad (13c)$$

Here N is the Brunt–Väisälä frequency, $g_i = (0, 0, g)$ is the acceleration vector due to gravity, α is the volume expansion coefficient, and $\partial T/\partial x_i$ is the mean potential temperature gradient.

If one assumes that $\epsilon(k)$ is inertial, $\epsilon(k) \equiv \epsilon$, the above procedure can easily be carried out with the well-known result

$$l_\epsilon = c_\epsilon^{-1} \Delta, \quad c_\epsilon = \pi \left(\frac{2}{3 \text{Ko}} \right)^{3/2}, \quad (14a)$$

where Ko is the Kolmogorov constant (≈ 1.6).

We shall employ Eq. (8a) with

$$l_\epsilon = c_\epsilon^{-1} l \quad (14b)$$

and express our result in terms of l with the condition that in the *neutral and shearless* case, $l = \Delta$.

As we have said, the lack of an a priori model to compute l has prompted suggestions of empirical expressions for l . Of these, the most widely used are the following.

In Deardorff (1980) the goal was to suppress a spurious heat flux within the stable inversion layer in the

planetary boundary layer (PBL) by decreasing l_ϵ with stratification. The suggested form of l was

$$l/\Delta = c \text{Fr} \quad (15)$$

with $c = 0.76$. Deardorff's model has been widely used in both SGS models and ensemble average models. On the other hand, Deardorff (1974, Fig. 19) himself had previously found from LES calculations that the dissipation length scale L_ϵ defined in terms of the total kinetic energy E (including all scales)

$$L_\epsilon = \frac{E^{3/2}}{\epsilon} \quad (16)$$

increases with stratification within the stable layer. Using LES data, Moeng and Wyngaard (1989) and Schmidt and Schumann (1989) confirmed that L_ϵ increases with stratification, and Schumann (1991) has commented that this increase of L_ϵ "contradicts the expectation which forms the basis of Deardorff's proposal." It was also found (Schumann 1990, 1991) that L_ϵ decreases with increasing importance of shear and becomes larger for buoyancy-dominated stable layers. The complex behavior of L_ϵ remained unexplained on physical grounds.

Hunt et al. (1988, 1989) suggested that for shear flow away from the boundaries, l_ϵ depends on S in the following way:

$$l_\epsilon = c \frac{\Delta}{\text{Sh}} \quad (17)$$

with $c = (3/2A_s)$ and $A_s = 0.46$.

In Canuto and Minotti (1993), the SGS model was based on the following physical argument. In the case of stable stratification and in the buoyancy subrange of the spectrum, the turbulent eddies work against gravity and lose a fraction of their turbulent kinetic energy (TKE), which is transformed into potential energy. Therefore, less TKE is available to be transferred to the inertial subrange and eventually to be dissipated by molecular effects. Furthermore, ϵ decreases faster than $e^{3/2}$ with stratification, corresponding to an increasing l_ϵ . For $\text{Fr}^{-1} < 4$, the model predicts

$$l = \Delta \exp(0.053 \text{Fr}^{-2}). \quad (18a)$$

The key assumption of this model is that the flux Richardson number R_f for the SGS scales is very large; that is, buoyancy dominates over shear. Here R_f is defined as the ratio between the destruction of TKE by buoyancy and the production of TKE by shear; namely,

$$R_f = \frac{g_i \alpha u_i \bar{\theta}}{\bar{u}_i u_j S_{ij}} = \frac{1}{\sigma_r} \frac{1}{(\text{Fr Sh})^2}, \quad (18b)$$

where σ_r is the turbulent Prandtl number. The problem was to decide whether production by shear or destruction by convection vanishes first in the small-scale SGS region of interest or, alternatively, what is their wave-

number dependence. It is clear, for example, that at very high wavenumbers, when the small-scale eddies are isotropic, the denominator in (18b) vanishes:

$$\overline{u_i u_j} S_{ij} \rightarrow e \delta_{ij} S_{ij} \rightarrow e S_{ii} \rightarrow 0 \quad (18c)$$

on account of the incompressibility condition. This suggests a large value of R_f (if $g_i \alpha u_i \theta$ decreases less fast). Stated differently, the assumption behind (18a) was that the production by shear vanishes at lower wavenumbers than buoyancy, so that by the time one reaches the SGS wavenumber region, buoyancy destruction is more important than shear production. Recent LES calculation by Kaltenbach (1992), however, show that, as a function of wavenumber, buoyancy production may actually vanish at lower wavenumbers than shear production, thus leading to a small rather than a large flux Richardson number.

The present paper addresses the problem of deriving the dissipation length scale l_ϵ in stably stratified turbulence as a two-parameter function $l_\epsilon(\text{Fr}, R_f)$. In section 2, we present the basic equations. In section 3, we discuss the effects of different timescales (Lagrangian or Eulerian) on l_ϵ . In section 4, we derive a length scale model based on the formulation of the buoyancy spectrum $B(k)$ of Shur (1962) and Lumley (1964). In section 5, we discuss Weinstock's formulation of $B(k)$ and give a corresponding length scale model. In section 6, we discuss our formulation of $B(k)$, which improves over both Lumley's and Weinstock's formulations, and derive a more complete model of the dissipation length scale l_ϵ . In section 7, we discuss the behavior of our expression for l_ϵ , and show how previous expressions for l_ϵ suggested by Deardorff, Hunt et al., Weinstock, and Canuto and Minotti can be recovered naturally from the more complete model. In section 8, we discuss the physical meaning of the reference length Δ . In section 9, we apply our model to a shear-driven PBL and derive l_ϵ as a function of height. We show that the result of our model closely reproduces the well-known LES result. An alternative form of the energy dissipation rate ϵ is discussed in section 10. Discussions on the model and its role in both ensemble average and LES calculations will be presented in section 11.

2. Basic equations of the model

Following Phillips (1965) and Weinstock (1978), we study the balance relation for the kinetic energy spectrum $E(k)$ in the buoyancy-inertia subranges in the wavenumber space under the assumptions that 1) the variation of $E(k)$ due to the transfer in physical space by the turbulence itself is negligible and 2) the contribution of the molecular term is negligible. The resulting equation is

$$\frac{\partial E(k)}{\partial t} = P(k) + B(k) - \frac{\partial \epsilon(k)}{\partial k}, \quad (19a)$$

where $P(k)$ and $B(k)$ are the spectra of shear and buoy-

ancy productions, respectively; $B(k) < 0$ for stably stratified flows. Lumley (1965) assumed that (Panofsky and Mares 1968; Wyngaard and Cote 1972)

$$P(k) \approx -\frac{1}{R_f} B(k), \quad (19b)$$

where R_f is defined in Eq. (18b). In order to parameterize the time variation of e at different Richardson numbers, Schumann (1992) introduced a growth-rate parameter in physical space. Similarly, we introduce a growth-rate parameter for $E(k)$ in spectral space

$$G = \frac{P(k)}{\partial \epsilon(k) / \partial k - B(k)} \quad (20a)$$

and leave it to be parameterized. Equations (19a)–(20a) yield

$$\frac{\partial \epsilon(k)}{\partial k} = \left(1 - \frac{1}{GR_f} \right) B(k). \quad (20b)$$

In the stationary limit, $G = 1$, while in neutral flows, $G > 1$. In Eq. (20b) we can rewrite G as

$$G = \frac{1}{R_{fc}}, \quad (21a)$$

where R_{fc} is a critical value of R_f , so that

$$\frac{\partial \epsilon(k)}{\partial k} = \left(1 - \frac{R_{fc}}{R_f} \right) B(k). \quad (21b)$$

For ensemble average models, R_f is always below R_{fc} in order to balance the e budget equation; for SGS models, however, R_f may exceed R_{fc} since additional kinetic energy may be cascaded from scales larger than the grid size.

Let us now consider the buoyancy spectrum $B(k)$. Since the physically relevant ingredients are the external timescale represented by the Brunt-Väisälä frequency N , the internal turbulence timescale τ , and the turbulent kinetic energy spectrum $E(k)$, we can write

$$B = B(N, \tau, E). \quad (22a)$$

As shown in appendix A,

$$B(k) = -N^2 \tau E(k). \quad (22b)$$

First, by its very nature, τ is a Lagrangian and not an Eulerian timescale. In spite of much work on the subject, calculations of Lagrangian variables are still the object of some controversy (Saffman 1963; Kraichnan 1970; Riley and Patterson 1974; Tennekes 1975; Yakhot et al. 1989; Chen and Kraichnan 1989; Nelkin and Tabor 1989; Gotoh and Kaneda 1991; Kaneda and Gotoh 1991; Gotoh et al. 1993). The general conclusion, however, is that in 3D turbulence, Lagrangian timescales are in general larger than Eulerian timescales and that the most appropriate definitions are as follows:

$$\tau_E^{-1} = k\epsilon^{1/2} \quad (23a)$$

$$\tau_L^{-1} = \left(\int_0^k p^2 E(p) dp \right)^{1/2} \quad (23b)$$

with

$$\tau_L > \tau_E. \quad (23c)$$

Second, whichever τ one chooses in Eq. (22b), one must account for one more fact: in the presence of stable stratification, τ is a function of the Brunt–Väisälä frequency (Weinstock 1978),

$$\tau = \tau(N). \quad (24)$$

The physical reason behind this assertion is that under conditions of stable stratification, eddies lose kinetic energy by working against gravity; the energy absorbed from the particle motion corresponds to spontaneous emission of a gravity wave.

3. The effect of the timescale τ on l_ϵ : Eulerian versus Lagrangian

In order to exhibit the effect on l_ϵ of choosing either τ_L or τ_E , we proceed as follows. Substitute Eq. (22b) into Eq. (21b). After using (10) and integrating, one obtains

$$\frac{\epsilon(k)}{\epsilon} = \left[1 + \frac{1}{3} \text{Ko} \epsilon^{-1/3} N^2 \left(1 - \frac{R_{fc}}{R_f} \right) \times \int_k^\infty \tau(p) p^{-5/3} dp \right]^3, \quad (25a)$$

which is valid for any τ . Furthermore, using Eqs. (9) and (8a), we derive

$$l_\epsilon^{2/3} = \text{Ko} \int_{\pi/\Delta}^\infty k^{-5/3} \left[\frac{\epsilon(k)}{\epsilon} \right]^{2/3} dk. \quad (25b)$$

Inspection of (25a) shows that if $R_f < R_{fc}$, the larger the τ , the smaller $\epsilon(k)/\epsilon$ will be, and conversely if $R_f > R_{fc}$. Because of (23c), we conclude from (25b) that

(i) $R_f < R_{fc}$

$$l_\epsilon(L) < l_\epsilon(E) \quad (26a)$$

(ii) $R_f > R_{fc}$

$$l_\epsilon(L) > l_\epsilon(E), \quad (26b)$$

where the symbols L and E stand for Lagrangian and Eulerian, respectively. Relations (26) indicate that although in general $\tau_L > \tau_E$, $l_\epsilon(L)$ may be larger or smaller than $l_\epsilon(E)$ depending on the value of the flux Richardson number.

4. Lumley's formulation of $B(k)$

Lumley (1964) neglected the τ dependence on N but accounted for the Lagrangian nature of τ . Using (23b) and (10), one derives

$$\tau_L \approx \left(\frac{3}{4} \text{Ko} \right)^{-1/2} \epsilon(k)^{-1/3} k^{-2/3} \quad (27a)$$

so that (22b) yields

$$B(k) = -2 \left(\frac{1}{3} \text{Ko} \right)^{1/2} N^2 \epsilon^{1/3}(k) k^{-7/3}, \quad (27b)$$

which is Lumley's result except for a numerical factor. The result is also derived in a more formal manner in appendix A. Using Eq. (27b) in (21b) and integrating from k to infinity with $\epsilon = \epsilon(\infty)$, one obtains

$$\epsilon(k) = \epsilon \left[1 + \left(\frac{1}{3} \text{Ko} \right)^{1/2} \left(1 - \frac{R_{fc}}{R_f} \right) \left(\frac{k_0}{k} \right)^{4/3} \right]^{3/2}, \quad (27c)$$

where k_0 is the Ozmidov wavenumber

$$k_0 = \left(\frac{N^3}{\epsilon} \right)^{1/2}. \quad (27d)$$

From Eqs. (10) and (27c), it follows that for $R_f \gg R_{fc}$,

$$E(k) \approx \begin{cases} k^{-3}, & k < k_0 \end{cases} \quad (28a)$$

$$\begin{cases} k^{-5/3}, & k > k_0, \end{cases} \quad (28b)$$

and thus k_0 separates the “buoyancy subrange” (28a) from the “inertial subrange” (28b).

Inserting (27c) into (10) and the result into (9), one can obtain relation (12a), which when cast in the form (8a), yields the following expression for the length l :

$$l = \Delta \left[1 - (2\pi^2\sqrt{3})^{-1} \text{Ko}^{3/2} \left(1 - \frac{R_{fc}}{R_f} \right) \text{Fr}^{-2} \right]^{-3/2}. \quad (28c)$$

The lack of the dependence on buoyancy of the timescale τ makes the application of this expression doubtful in the case of strong stratification. In the case of weak stratification, however, the formula ought to apply. In fact, in the case

$$N^2 \rightarrow 0, R_f \rightarrow 0, \quad (28d)$$

use of expression (18b)

$$R_f = \frac{1}{\sigma_t (\text{Fr Sh})^2} \quad (28e)$$

yields

$$l/\Delta = [1 + (2\pi^2\sqrt{3})^{-1} \text{Ko}^{3/2} R_{fc} \sigma_{t0} \text{Sh}^2]^{-3/2}, \quad (28f)$$

where σ_{t0} is the turbulent Prandtl number in the absence of stratification. The same result is also derived from our more complete model in the neutral limit [see Eq. (41a)]. It will be shown later that for intermediate values of Sh, Eq. (28f) reproduces the empirical expression of Hunt et al. (1988, 1989).

In conclusion, even when Lumley's original model is extended to include R_f , the final expression for l_ϵ is acceptable only in the case of weak stratification.

5. Weinstock's formulation of $B(k)$

Weinstock (1978) assumed Eq. (23a) but accounted for the N dependence (24). The choice of an Eulerian timescale is discussed in appendix A. The main contribution of Weinstock's work is to have pointed out the importance of buoyancy effects on the timescale τ . Indeed, a recent application of a second-order closure model to the PBL has shown that the use of $\tau = \tau(N)$ considerably improves the behavior of the dissipation ϵ when compared with LES data (Canuto et al. 1994). Since the intermediate steps to derive the $\tau = \tau(N)$ dependence are somewhat complicated, they are reproduced in their essence in appendix A, while the details can be found in the original paper. The final result, however, is simply represented by the fact that, as expected, buoyancy affects the turbulence timescale τ in such a way that

$$\tau \rightarrow \frac{\tau}{1 + \tau^2 N^2}. \quad (29a)$$

The effect of buoyancy is that of adding the term $\tau^2 N^2$ in the denominator, which physically represents "energy absorbed from the particle motion" (Weinstock 1978), a damping factor quite common in similar situations in plasma physics. Using Eq. (23a), we obtain from (29a) that

$$\tau = \frac{e^{1/2} k}{ek^2 + N^2}, \quad (29b)$$

which is indeed Weinstock's expression except for a numerical coefficient (6/5) in front of N^2 . Thus, the buoyancy spectrum (22b) has the form

$$B(k) = -(3/2)^{1/2} a N^2 E(k) e^{1/2} k \left[ek^2 + \frac{6}{5} N^2 \right]^{-1}, \quad (30)$$

where a is an anisotropy factor to be discussed later.

Inserting now Eq. (30) into (21b) and carrying out the integration yields

$$\epsilon(k) = \epsilon \left[1 + (5/6)^{1/2} \text{Ko} \times \left(1 - \frac{R_{fc}}{R_f} \right) \left(\frac{k_0}{k_B} \right)^{2/3} C(k/k_B) \right]^3, \quad (31a)$$

where

$$k_B \Delta = (6/5)^{1/2} \text{Fr}^{-1}. \quad (31b)$$

The k dependence of $\epsilon(k)$ is entirely contained in the function $C(x)$, defined as

$$C(x) = \int_x^\infty a(\eta) \frac{\eta^{-2/3}}{1 + \eta^2} d\eta. \quad (31c)$$

Based on some earlier experiments on tropospheric and stratospheric spectra, Weinstock (1978) approximated the anisotropy factor a in Eq. (30) as

$$a(k/k_B) = \begin{cases} \frac{1}{2}, & k \leq k_B \\ 1, & k \geq k_B, \end{cases} \quad (31d)$$

which has a discontinuity at $k = k_B$. We propose instead the following form:

$$a(k/k_B) = \begin{cases} k/k_B, & k \leq k_B \\ 1, & k \geq k_B, \end{cases} \quad (31e)$$

since it is the simplest form that reflects the fact that $a(k/k_B)$ continuously decreases with decreasing k/k_B for $k \leq k_B$.

Following the procedure outlined before, we substitute Eq. (31a) into Eqs. (10) and (9) to obtain Eq. (12a), which is then cast in the form of Eq. (8a). The result is Eq. (B2) of appendix B. Since such an expression is too unwieldy, we have devised a simpler expression that approximates very well the result (B2):

$$l_\epsilon = c_\epsilon^{-1} l, \quad (32a)$$

$$\frac{l}{\Delta} = \left[\left(1 - \frac{\Omega^2}{(a_2 + \text{Fr}^2)^2} \right)^{1/2} - \frac{\Omega}{a_3 + \text{Fr}^2} \right]^{-3} \quad (32b)$$

$$\Omega \equiv a_1 \left(1 - \frac{R_{fc}}{R_f} \right) \quad (32c)$$

$$a_1 = 8.68 \times 10^{-3} \text{Ko}^{3/2}, \quad a_2 = 0.025, \quad a_3 = 0.014, \quad (32d)$$

where Fr is the Froude number defined in Eq. (13b).

In the limit of weak stratification, $N^2 \rightarrow 0$, Eq. (32b) yields, using the second of (18b),

$$\frac{l}{\Delta} = [(1 - p^2 \text{Sh}^4)^{1/2} + p \text{Sh}^2]^{-3} \quad (33a)$$

$$p \equiv a_1 R_{fc} \sigma_{i0}, \quad (33b)$$

where Sh is the shear number defined in Eq. (13b), and σ_{i0} is the value of the turbulent Prandtl number in the neutral limit. Equation (33a) can be compared with Eq. (28f), which is based on Lumley's $B(k)$. Clearly, (33a) is limited to shear number less than

$$\text{Sh} < p^{-1/2}. \quad (33c)$$

In this region of validity, it is easy to verify that Eqs. (33a) and (28f) satisfy the general relation (26a).

In the presence of buoyancy, the length scale model Eq. (32b) based on Weinstock's $B(k)$ can be more realistic than Eq. (28c) based on Lumley's $B(k)$, since (32b) accounts for the important buoyancy effect on the timescale, while (28c) does not. In the absence of buoyancy, on the other hand, (28c) can be more physical than (32b) since (28c) is based on a Lagrangian

timescale, while (32b) is based on an Eulerian timescale.

6. New model

Starting from the buoyancy spectrum expression, Eq. (22b), we have presented two models of l_ϵ for stably stratified turbulence. In the first model (section 4), a Lagrangian timescale was employed in Eq. (22b), but the effect of buoyancy on the timescale was not considered; in the second model (section 5), the effect of buoyancy was appropriately accounted for, but the timescale employed was Eulerian. Neither model is thus complete. A model that includes both τ_L and the stratification dependence of τ_L will be introduced below.

Under the approximation of local inertiality, the Lagrangian timescale defined in Eq. (23b) becomes

$$\tau_L = \left(\frac{3}{4} \text{Ko} \right)^{-1/2} \epsilon(k)^{-1/3} k^{-2/3}. \quad (34)$$

We rewrite Eq. (29a), which has been designed to include buoyancy effect on the timescale, as

$$\tau = \frac{\tau_L}{1 + \tau_L^2 N^2}. \quad (35)$$

Then we insert Eqs. (34), (35), and (22b) into Eq. (21b). The resulting differential equation can be integrated analytically with the following result:

$$\epsilon(k) = \epsilon \left[1 + \frac{4}{3 \text{Ko}} \Omega N^2 \epsilon(k)^{-2/3} k^{-4/3} \right]^{-\xi}, \quad (36a)$$

where

$$\xi = \frac{3}{2} (1 - \Omega^{-1}),$$

$$\Omega = 1 + \frac{3^{1/2}}{4} \text{Ko}^{3/2} \left(\frac{R_{fc}}{R_f} - 1 \right). \quad (36b)$$

Now we use Eqs. (36a–b) in Eq. (10) and then integrate Eq. (9). The resulting e is then used in Eq. (8a). The ensuing expression for l_ϵ is

$$l_\epsilon = c_\epsilon^{-1} l, \quad (37a)$$

where

$$\left(\frac{l}{\Delta} \right)^{2/3} \equiv \int_0^1 f dy \quad (37b)$$

$$f = \left[1 + 2\pi^{-2} \Omega \text{Fr}^{-2} \left(\frac{l}{\Delta} \right)^{2/3} f^{-1} y^2 \right]^{-2\xi/3}. \quad (37c)$$

Note that the right side of Eq. (37c) contains both l/Δ and f itself. Therefore, l/Δ can only be solved numerically from Eqs. (37b) and (37c). We have nonetheless

found that the following algebraic expression approximates very closely the exact solution:

$$\frac{l}{\Delta} = [1 + a \text{Fr}^{-2} (1 + b \text{Fr}^{-4/3})^{-1}]^{-c}, \quad (38a)$$

where

$$R_f < R_{fc}: \quad a = 2(3\pi^2)^{-1}(\Omega - 1), \\ b = 0.12 \left(\Omega - 1 + \frac{3}{2} \Omega^{-1} \right)^{4/9}, \quad c = \frac{3}{2} \quad (38b)$$

$$R_f > R_{fc}: \quad a = 4(5\pi^2)^{-1}\Omega, \quad b = 0, \\ c = \frac{5}{4} (1 - \Omega^{-1}), \quad (38c)$$

and Ω is given in Eq. (36b). The comparison between Eqs. (37b) and (38a) is presented in Fig. 1. Since (38a) reproduces (37b) very well in wide ranges of the two parameters (Fr , R_f), we suggest the use of (38a) as the new dissipation length scale formula. The new, two-parameter (Fr , R_f) model reflects the combined effects of buoyancy (sink of short scales) and shear (source of short scales) on the TKE dissipation length scale.

In summary, the new model predicts that (Fig. 1)

$$\frac{l}{\Delta} \begin{cases} \text{decreases} & \text{if } R_f < R_{fc} \\ \text{increases} & \text{if } R_f > R_{fc} \end{cases} \quad \text{with stratification} \quad (39)$$

7. Recovery of previous models

Previous formulas for l_ϵ or ϵ by Deardorff, Hunt et al., Weinstock, and Canuto and Minotti can be shown to be special cases of the present model.

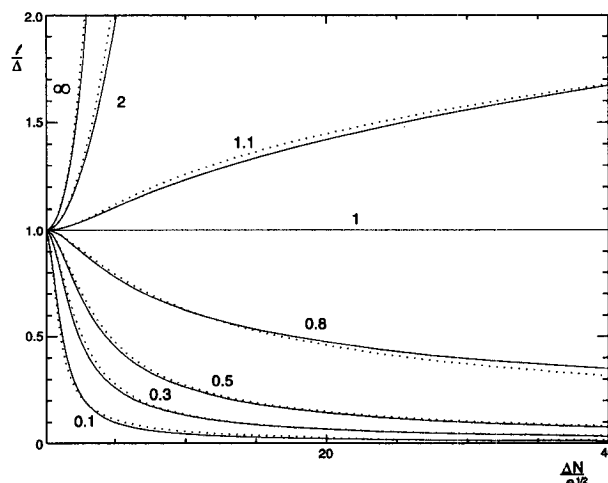


FIG. 1. The normalized dissipation length scale l/Δ according to the present model equations (37b–c) (solid lines), and according to the algebraic equation (38a) (dotted lines), versus the inverse Froude number $\text{Fr}^{-1} = \Delta N \epsilon^{-1/2}$, for the normalized flux Richardson number $R_f/R_{fc} = 0.1, 0.3, 0.5, 0.8, 1, 1.1, 2$, and ∞ .

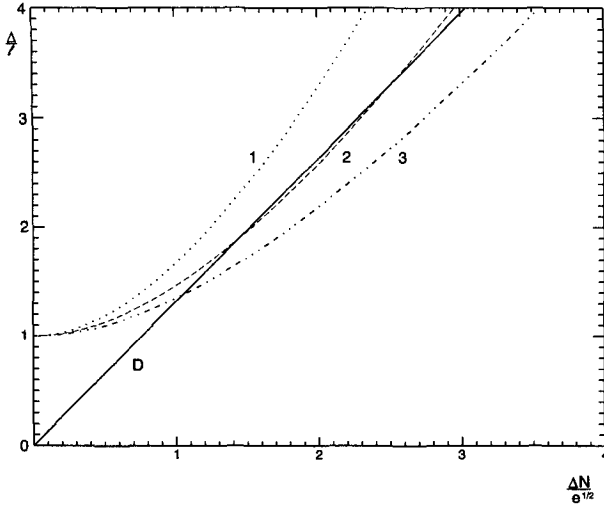


FIG. 2. Plot of Δ/l vs $Fr^{-1} = \Delta Ne^{-1/2}$. Curve 1: Eq. (38a) with $R_f/R_{fc} = 0.1$; curve 2: Eq. (38a) with $R_f/R_{fc} = 0.14$; curve 3: Eq. (38a) with $R_f/R_{fc} = 0.18$; curve D: Eq. (15), Deardorff's model (1980).

a. Deardorff's formula (1980)

For $R_f < R_{fc}$ and large Fr^{-1} , the model equation (38a) can be approximated as

$$\frac{\Delta}{l} \approx (c Fr)^{-1}, \quad (40a)$$

where c is dependent on R_f/R_{fc} and Ko . Equation (40a) indicates that Δ/l increases linearly with Fr^{-1} . This is in agreement with Deardorff's (1980) expression, Eq. (15):

$$\frac{\Delta}{l} = (c Fr)^{-1}, \quad c = 0.76. \quad (40b)$$

While Deardorff's constant c has no formal justification, however, the present model suggests that c is a function of R_f/R_{fc} and Ko . For $Ko = 1.6$, $R_f/R_{fc} \approx 0.15$, and for $1 < Fr^{-1} < 10$, the two constants nearly coincide. In Fig. 2 we plot Δ/l versus Fr^{-1} according to both Deardorff's model and the present model, Eq. (38a). Furthermore, it must be noted that for small buoyancy ($Fr^{-1} \rightarrow 0$), Deardorff's l/Δ diverges, while the present model predicts $l \rightarrow \Delta$, which is the correct physical limit.

b. Hunt et al.'s formula (1988, 1989)

In the neutral limit ($Fr^{-1} \rightarrow 0$, $R_f \rightarrow 0$), Eq. (37b) or (38a) reduces to

$$\frac{l}{\Delta} = (1 + p Sh^2)^{-3/2} \quad (41a)$$

$$p = (2\sqrt{3}\pi^2)^{-1} Ko^{3/2} R_{fc} \sigma_{i0}, \quad (41b)$$

where Sh is the shear number defined in Eq. (13b) and σ_{i0} is the neutral value of the turbulent Prandtl number. In Fig. 3, we plot Δ/l according to Eq. (41a) together with Hunt et al.'s model, Eq. (17). For intermediate values of Sh , the present model result nearly coincides with that of Hunt et al. However, as $Sh \rightarrow 0$, the Hunt et al. l/Δ diverges, while in the present model $l/\Delta \rightarrow 1$, as expected.

c. Weinstock's formula (1981)

For $R_f \approx 0.5 R_{fc}$, one can estimate from Fig. 1 that

$$\Delta/l \approx 0.4 Fr^{-1}, \quad (42a)$$

which, using Eqs. (8a) and (14b), leads to

$$\epsilon \approx 0.4 c_e N \approx 0.5 \overline{w^2} N. \quad (42b)$$

In the last step, we have approximated e by $3/2 \overline{w^2}$. Equation (42b) was analytically derived by Weinstock (1981) for stably stratified flows and is consistent with stratospheric data.

d. Canuto and Minotti's formula (1993)

For $R_f \gg R_{fc}$ the present model, Eq. (38a), reduces to (for $Ko = 1.6$)

$$\frac{l}{\Delta} = [1 + 0.01 Fr^{-2}]^{8.9}. \quad (43a)$$

For small Fr^{-1} , (43a) further reduces to

$$\frac{l}{\Delta} = 1 + 0.089 Fr^{-2}. \quad (43b)$$

This result of l/Δ should be compared with Eq. (18a), which was based on an Eulerian timescale. We can no-

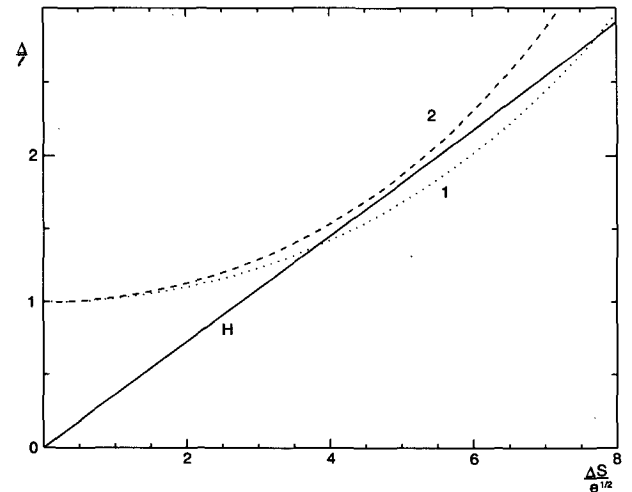


FIG. 3. Plot of Δ/l vs $Sh = \Delta Se^{-1/2}$ in the neutral limit. Curve 1: Eq. (41a) with $R_{fc} = 0.4$; curve 2: Eq. (41a) with $R_{fc} = 0.5$; curve H: Eq. (17), Hunt et al.'s model (1988, 1989).

tice that for $Fr^{-1} < 4$, (43b) and (18a) satisfy the general relation (26b).

8. The reference length Δ

In SGS models, Δ is the grid size corresponding to the smallest resolved scale defined in Eq. (9). In Deardorff's formulation, Eq. (15), Δ was referred to as the length scale under *neutral condition*, but we have seen from our discussion that it is more appropriate to refer to Δ as the dissipation length scale under *neutral and shearless conditions*. Under these conditions, the TKE simply cascades through a purely inertial range to the dissipation scales. The length scale l is equal to Δ only under neutral and shearless conditions. In general, however, when there is buoyancy and/or shear, l does not equal Δ , but is given by Eq. (38a).

In ensemble average models, following the argument given by Lumley (1964), we assume that the Reynolds number is so large that the spectrum $E(k)$ can be approximated as rising sharply from zero at π/Δ . In other words, the buoyancy-inertia subranges can be regarded as having been extended to the largest scales. Here Δ is, in general, a function of space coordinates and may be affected by the boundaries and by nonlocal phenomena. The main purpose of the present model is to give the dependence of l on buoyancy and shear, or, in other words, on how far and along which direction l deviates from Δ because of buoyancy and shear forcing; Δ itself needs to be prescribed from outside the model (the same occurs in Deardorff and Hunt et al.'s models). In the planetary boundary layer (PBL), Δ is of course a function of height.

9. Application of the present model to the PBL

Recent work of Moeng and Sullivan (1994) provides us with the latest LES data in a shear-driven PBL with a capping inversion. These data are ideal to test the new model of l , since both the turbulent transport and the pressure transport (in physical space) are negligibly small in the shear-driven PBL. The LES data give the vertical profiles of the mean potential temperature $T(z)$, the TKE $E(z)$ as contributed by all the scales, the shear production $P(z)$, and the buoyancy production $B(z)$. From these data, we constructed the flux Richardson number $R_f(z) = -B(z)/P(z)$ and the Froude number $Fr(z) = E(z)^{1/2}/[\Delta(z)N(z)]$, where $N(z) = [gT(z)^{-1}dT(z)/dz]^{1/2}$ is the Brunt-Väisälä frequency and $\Delta(z)$ will be prescribed in the caption of Fig. 4.

Substituting the resulting $Fr(z)$ and $R_f(z)$ into the present model [Eq. (38a)], we obtain l/Δ as a function of $[Fr(z), R_f(z)]$, which is plotted as a two-parameter diagram in Fig. 4, where the points A, B, . . . and I denote successive heights from the ground in the PBL. From Fig. 4 one can see that l/Δ is a nonmonotonic function of height.

Using the same LES data for the shear-driven PBL, we constructed a dissipation length L_ϵ defined as in Eq. (16),

$$L_\epsilon(\text{LES}) = E^{3/2}(z)/\epsilon(z). \quad (44a)$$

On the other hand, using the model equation (38a), as we have done in Fig. 4, we also constructed $L_\epsilon(\text{model})$ defined as follows:

$$L_\epsilon(\text{model}) = c_\epsilon^{-1}l[Fr(z), R_f(z)], \quad (44b)$$

where the input parameters $Fr(z)$ and $R_f(z)$ were again obtained from the same LES data. In Fig. 5 we plot both $L_\epsilon(\text{LES})$ and $L_\epsilon(\text{model})$, normalized by the PBL depth Z_i , versus z/Z_i . As one can see, our model reproduces rather closely the LES vertical profile of L_ϵ .

Neither Deardorff's model nor the Hunt et al. model can explain the LES L_ϵ versus z curve, whose non-monotonic behavior has been found in all LES calculations, beginning with the one by Deardorff (1974, Fig. 19) and more recently by, for example, Moeng and Wyngaard (1989) and Schmidt and Schumann (1989), as well as by field measurements (Jochum et al. 1990). The underlying reason for this failure is that both models tried to determine the behavior of L_ϵ using only one variable, either Fr or Sh . On the other hand, we have seen that our model, Eq. (38a), calls for the presence of two variables, Fr and R_f (or Fr and Sh). The physical explanation of the complex profile of L_ϵ offered by the

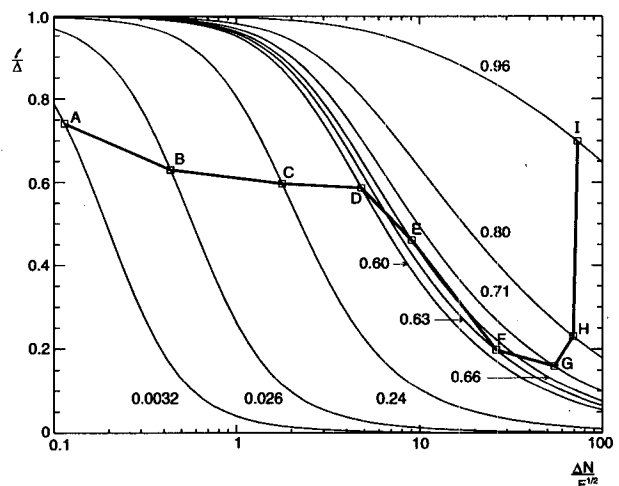


FIG. 4. Plot of l/Δ , as given by Eq. (38a), vs $\Delta N E^{-1/2}$ for different values of $R_f/R_{fc} = 0.0032, 0.026, 0.24, 0.6, 0.63, 0.66, 0.71, 0.8$, and 0.96 . Superimposed are nine representative points (A, B, . . . , and I) in order of increasing z from the ground. The locations of these points in the $(Fr^{-1}, R_f/R_{fc})$ plane were determined using the LES data of Moeng and Sullivan (1994) discussed in the text. The value of R_{fc} is determined as 0.43 in accordance with Fig. 3. The reference length Δ in the definition of Fr is a function of height z (section 8) and is simply prescribed as $\Delta = 3z$ for $z/Z_i \leq 0.4$ and $\Delta = \text{constant}$ for $z/Z_i \geq 0.4$, where Z_i is the PBL depth. This figure shows how l deviates from Δ due to the combined effects of buoyancy and shear and exhibits a nonmonotonic behavior as a function of height in the PBL.

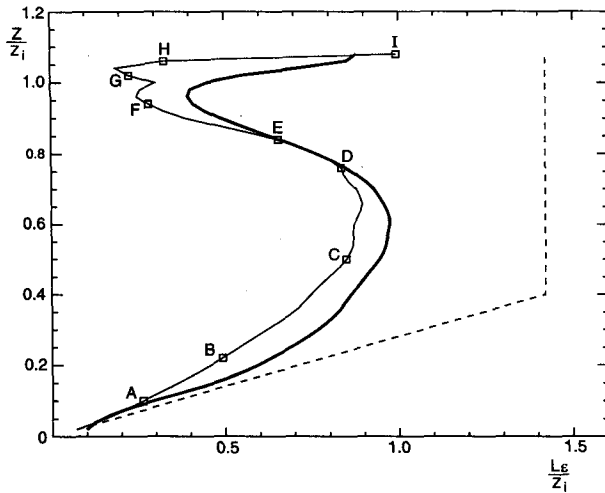


FIG. 5. This figure refers to the same shear-driven PBL as in Fig. 4, and the letters A, B, C, . . . , and I denote the same successive heights as in Fig. 4. Here l_e , normalized by the PBL depth Z_i , is plotted as a function of the normalized height z/Z_i according to the present model [Eq. (38a), thin solid line] and according to the LES result (thick solid line). The S-shaped vertical profile of l_e , which has been evident from LES calculations and field measurements, is well reproduced by the present model. Also plotted is the reference length scale Δ (normalized by $c_i Z_i$, dashed line).

present model can be seen clearly by comparing Fig. 4 and Fig. 5: Eq. (38a) tells that L_e decreases as Fr^{-1} increases if $R_f < R_{fc}$ is fixed; on the other hand, L_e increases as R_f increases if Fr^{-1} is fixed. Since in the PBL, both R_f and Fr^{-1} vary with height, L_e may decrease or increase depending on the combined effect of Fr^{-1} and R_f (the points A, B, . . . , I represent successive heights from the ground in both Figs. 4 and 5).

10. An alternative form for ϵ

It is possible to obtain an alternative form of the dissipation rate ϵ that does not directly involve the turbulent kinetic energy e . Integrating Eq. (21b) from $k = k_m = \pi/\Delta$ to $k = \infty$, we obtain, with the aid of Eq. (A1),

$$\epsilon - \epsilon(k_m) = \left(1 - \frac{R_{fc}}{R_f}\right) g \alpha \overline{w\theta}; \quad (45)$$

$\epsilon(k_m)$ may be obtained by solving Eq. (36a) iteratively, as a function of N , Δ and $R_f R_{fc}^{-1}$. Thus, ϵ according to Eq. (45) may be expressed as a function of N , Δ , $R_f R_{fc}^{-1}$, and the vertical heat flux $w\theta$, so it may be useful in those turbulence models where $w\theta$ is given or solved prognostically.

11. Summary and discussion

We have presented a new model for the turbulent kinetic energy dissipation length scale l_e . The model

was derived using the solution of the turbulent kinetic energy spectral equation, Eq. (21b). The buoyancy spectrum entering such equation was constructed using both a Lagrangian timescale and the effect on it of stable stratification. Both shear production and buoyancy destruction in the buoyancy-inertia subranges were considered. The model contains substantial information concerning the dependence of the length scale l_e on the combined effects of shear and buoyancy. The main results can be summarized as follows.

- 1) For $R_f < R_{fc}$, stratification and shear (separately) decrease l_e .
- 2) For $R_f < R_{fc}$, the model embodies Deardorff's expression as a special case.
- 3) For $R_f \approx 0$, the model embodies the Hunt et al. expression as a special case.
- 4) While Deardorff's model considers only buoyancy and the Hunt et al. model considers only shear and both models are not valid for small buoyancy or shear, the present model considers the combined effects of buoyancy and shear. The length scale in the present model naturally approaches its physical limit, Δ , as both buoyancy and shear vanish, whereas in the Deardorff or Hunt et al. model, it does not.
- 5) For larger values of R_f ($\approx 0.5 R_{fc}$), the present model encompasses Weinstock's analytic expression for ϵ .
- 6) For $R_f > R_{fc}$, a situation which may occur in SGS models, l_e increases with stratification; for $R_f \gg R_{fc}$, the model encompasses (qualitatively) Canuto and Minotti's model.

An alternative form of the dissipation rate ϵ [Eq. (45)] is also discussed that may be useful when the vertical heat flux is solved prognostically.

The present formulation of the dissipation length scale applies to both SGS models and ensemble average models. We applied the new model to the PBL and found that the dissipation length scale obtained from the model reproduces the nonmonotonic behavior of the length scale exhibited by all LES calculations since 1974, as well as by recent field measurements.

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APPENDIX A

Derivation of Eq. (22b)

The definition of the buoyancy spectral function $B(k)$ is defined so that (Weinstock 1978)

$$\int B(k) dk = g\alpha w\bar{\theta}, \quad (\text{A1})$$

where w is the vertical velocity fluctuation and θ is the temperature fluctuation. By definition,

$$B(k) \sim g\alpha k^2 \int d\Omega \int d\mathbf{r}' \overline{w(\mathbf{r}', t)\theta(\mathbf{r}, t)} \times \exp(-i\mathbf{k} \cdot \boldsymbol{\rho}), \quad (\text{A2})$$

where $d\Omega$ is the solid angle and $\boldsymbol{\rho} \equiv \mathbf{r} - \mathbf{r}'$. Since by integrating the temperature equation one has that θ is given by (T is the mean temperature)

$$\theta(\mathbf{r}, t) = -\frac{\partial T}{\partial z} \int_0^t dt' w[\mathbf{R}(t'), t-t'] \quad (\text{A3})$$

$$\mathbf{R}(t) = \mathbf{r} + \int_0^t dt' \mathbf{v}(\mathbf{R}(t'), t'), \quad (\text{A4})$$

one has to contend with the ‘‘Lagrangian trajectory’’ $\mathbf{R}(t)$ of a particle at time t , given that the particle was at the position \mathbf{r} at time zero. The correlation $w\bar{\theta}$ that enters (A2) can be written, with the help of (A3), as

$$\overline{w(\mathbf{r}', t)\theta(\mathbf{r}, t)} = -\frac{\partial T}{\partial z} \int_0^t \int d\mathbf{r}_0 dt' \overline{w(\mathbf{r}', t)w(\mathbf{r}_0, t-t')\delta[\mathbf{R}(t') - \mathbf{r}_0]} \quad (\text{A5})$$

or

$$\overline{w(\mathbf{r}', t)\theta(\mathbf{r}, t)} = -\frac{\partial T}{\partial z} R_L(t), \quad (\text{A6})$$

where R_L is the Lagrangian correlation function. At this point, one makes the so-called independent approximation (Corrsin 1960; Saffman 1963) whereby

$$R_L(t) = \int_0^t \int d\mathbf{r}_0 dt' \overline{w(\mathbf{r}', t)w(\mathbf{r}_0, t-t')\delta[\mathbf{R}(t') - \mathbf{r}_0]}. \quad (\text{A7})$$

Next, one employs the standard definition of the Eulerian spectrum $S(\mathbf{k}, t)$; namely,

$$S(k, t) = \int d\boldsymbol{\rho} \overline{w(\mathbf{r}', t)w(\mathbf{r}_0, t-t')} \exp(-i\mathbf{k} \cdot \boldsymbol{\rho}), \quad (\text{A8})$$

where

$$S(\mathbf{k}, t) \sim \overline{w^*(\mathbf{k}, 0)w(\mathbf{k}, t)} = \overline{w^*(\mathbf{k}, 0)w(\mathbf{k}, 0)} \times \Phi(k, t) \sim k^{-2}E(k)\Phi(k, t). \quad (\text{A9})$$

Combining the above expressions, one obtains that Eq. (A2) has indeed the form (22b) where τ is the result of a complicated integration

$$I \equiv \int dt' \Phi(k, t') \langle \exp[i\mathbf{k} \cdot \Delta\mathbf{R}(t')] \rangle, \quad (\text{A10})$$

where

$$\langle \exp[i\mathbf{k} \cdot \Delta\mathbf{R}(t)] \rangle = -\int d\boldsymbol{\rho} \langle \delta[\mathbf{R}(t) - \mathbf{r}_0] \rangle \exp(i\mathbf{k} \cdot \boldsymbol{\rho}), \quad (\text{A11})$$

where $\boldsymbol{\rho} = \mathbf{r} - \mathbf{r}_0$ (for ease of notation in dealing with exponentials, we have changed the notation from \bar{A} to $\langle A \rangle$).

The physical difference between the Shur–Lumley and Weinstock models lies in the evaluation of the integral I . If the function $\Phi(k, t)$ is taken from EDQNM (Lesieur 1991), we have

$$\Phi(k, t) \sim e^{-\eta(k)t} \sim e^{-t/\tau}, \quad (\text{A12})$$

with

$$\tau^{-1} \sim \eta(k) \approx \nu_t(k)k^2, \quad (\text{A13})$$

where $\nu_t(k)$ is the turbulent viscosity. Since

$$\nu_t(k) \approx \epsilon^{1/3}(k)k^{-4/3}, \quad (\text{A14})$$

it follows that

$$\tau^{-1} \sim \epsilon^{1/3}(k)k^{2/3}, \quad (\text{A15})$$

which we recognize as a Lagrangian timescale by inspection of Eq. (23b). Finally,

$$B(k) \sim k^2 C(k) \approx E(k)I(k) \sim k^{-2}E(k)\nu_t^{-1}(k) \sim \epsilon^{1/3}(k)k^{-7/3}, \quad (\text{A16})$$

which is the Shur–Lumley result (27b).

On the other hand, Weinstock (1978) did not employ (A12) but Kraichnan’s (1959) result (see also Saffman 1963)

$$\Phi(k, t) \approx e^{-1/2\langle [\mathbf{k} \cdot \Delta\mathbf{R}(t)]^2 \rangle}, \quad (\text{A17})$$

which he generalized to include buoyancy as follows:

$$\Phi(k, t) \approx e^{i\omega t} e^{-1/2\langle [\mathbf{k} \cdot \Delta\mathbf{R}(t)]^2 \rangle}, \quad (\text{A18})$$

where the frequency ω is a solution of a nonlinear dispersion relation.

Before we discuss the buoyancy dependence due to ω , we recall that the main reason Kraichnan’s DIA did not yield the Kolmogorov spectrum $k^{-5/3}$ but rather $k^{-3/2}$ is precisely because of the use of an Eulerian rather than a Lagrangian timescale. Since Weinstock adopted Kraichnan’s result for Φ , the lack of Lagrangian timescale persists in his calculations.

The major contribution of Weinstock’s work is that he pointed out the importance of the buoyancy effect on τ and carried out an explicit calculation. Physically, it seems clear that ω in Eq. (A18) ought to be of the order of the Brunt–Väisälä frequency N , and indeed the more exact result is

$$\omega = \omega_g = \frac{k_x}{k} N, \quad (\text{A19})$$

where ω_g is the gravity wave frequency and k_x is the horizontal component of k . In principle, one could write the function $\Phi(k, t)$ in the general form

$$\Phi(k, t) \approx e^{i\omega t} e^{-ik_z z}, \quad (\text{A20})$$

where k_z is the vertical component of k and z is the height. In the case of gravity waves, ω is real, while the wavenumber k_z is complex $k_z \rightarrow \text{Re}k_z + i\beta$; in the absence of nonlinear damping, $\beta = 1/(2H)$, where H is the pressure scale height. Thus, one concludes that gravity waves grow in space as $\exp(z/2H)$, a well-known result. If, on the other hand, one accounts for nonlinear damping, one can achieve saturation, $\beta = 1/2H - (\text{nonlinear damping}) = 0$. This holds true in the absence of turbulence. When the latter is included, the frequency ω is no longer real but

$$\omega = \omega_g + id, \quad (\text{A21})$$

where the damping factor d is responsible for the exponential in (A17).

Using the fact that for an approximately Gaussian random field, the exponential entering Eq. (A17) can be written as

$$\langle \exp[i\mathbf{k} \cdot \Delta \mathbf{R}(t)] \rangle \approx \exp \left\{ -\frac{1}{2} \langle [\mathbf{k} \cdot \Delta \mathbf{R}(t)]^2 \rangle \right\}; \quad (\text{A22})$$

the final form of the integral I is given by

$$\int dt' \exp \{ i\omega_g t' - \langle [\mathbf{k} \cdot \Delta \mathbf{R}(t')]^2 \rangle \} \approx \left(\frac{3}{2} \right)^{3/2} \frac{e^{-1/2} k^{-1}}{1 + (6/5)N^2/ek^2}. \quad (\text{A23})$$

Using now the definition (23a), we obtain that the right-hand side of (A23) can be written as

$$\tau \equiv \frac{\tau_E}{1 + (N\tau_E)^2}, \quad (\text{A24})$$

which shows that Weinstock's timescale fully accounts for the dependence on buoyancy but uses an Eulerian timescale τ_E .

As pointed out by a reviewer, Weinstock made a Lagrangian correction to the timescale, but in a different context (Weinstock 1991). He suggested that Eq. (A23) be modified to account for the Lagrangian effect (1993, personal communication).

APPENDIX B

Derivation of Eq. (32b)

In analogy with Weinstock (1978), for ease of computation, we first approximate the function $C(\zeta)$ in Eq. (31c) as

$$C(\zeta) = \begin{cases} \left(\frac{3}{4} \right) \frac{1 - \zeta^{4/3}}{1 + 0.2\zeta^2} + \frac{1}{2}, & \zeta < 1 \\ \left(\frac{3}{5} \right) \frac{\zeta^{-5/3}}{1 + 0.2\zeta^{-2}}, & \zeta > 1. \end{cases} \quad (\text{B1})$$

Using Eq. (B1) we substitute Eq. (31a) into Eq. (10) to find the energy spectrum $E(k)$ and then integrate $E(k)$ from $K_m = \pi/\Delta$ to infinity to find e according to Eq. (9). The expression of e so obtained can be cast in the form of Eq. (8a). After some algebra we obtain

$$l_e = c_e^{-1} l, \quad \frac{l}{\Delta} = [(1 - \delta_2/\delta_1)^{1/2} - \delta_1^{-1/2} \text{sgn}(1 - R_{fc}/R_f)]^{-3} \quad (\text{B2})$$

$$\delta_1 = \left(\frac{18}{5} \right)^2 \text{Ko}^{-3} (1 - R_{fc}/R_f)^{-2} \zeta_m^{-2/3} I_1^{-2}(\zeta_m) \quad (\text{B3})$$

$$\delta_2 = \frac{3}{2} \zeta_m^{-2/3} I_1^{-2}(\zeta_m) I_2(\zeta_m) - 1 \quad (\text{B4})$$

$$\zeta_m \equiv k_m/k_B = \left(\frac{5}{6} \right)^{1/2} \pi e^{1/2}/(N\Delta) = 2.87 \text{ Fr} \quad (\text{B5})$$

$$I_n(\zeta_m) = \int_{\zeta_m}^{\infty} C^n(\eta) \eta^{-5/3} d\eta. \quad (\text{B6})$$

Equations (B1)–(B6) form an expression for the turbulent kinetic energy dissipation length scale l_e based on Weinstock's formulation of the buoyancy spectrum. Aided by asymptotic analysis of the integrals I_1 and I_2 in Eq. (B6), a much simpler algebraic expression that fits Eq. (B2) very well is found and presented in the text as Eq. (32b).

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